Quadrotors Trajectory Tracking using a Differential Flatness-Quaternion based Approach

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Abstract— **A quadrotors is a type of Unmanned Aerial Vehicles (UAV) systems that attract the researchers in the control field since it's a highly nonlinear, underactuated system. In this paper, a non-linear dynamic model based on quaternions is developed. Differential flatness is an approach that enables the optimization to occur within the output space and therefore simplifies the problem of the trajectory tracking. The aim of this work is to create a Differential flatness-quaternion approach that enables the quadrotors to follow a desired path. The trajectory tracking is assured by a double loop control structure based on the LQR controller.**

Keywords—quadrotors; trajectory tracking; LQR; differential flatness; quaternions.

II. INTRODUCTION

Nowadays quadrotors have a growing interest from researchers comparing to the other types of Unmanned Aerial Vehicles (UAV) due to their several applications in both military and civil fields.

Many works have been published on control issues but the most recent converge over the modeling with quaternions instead of Euler angles to overcome the discontinuities, many control laws have been applied such as PID controllers, linear quadratic LQR algorithm, feedback linearization, and backstepping [1–5]. Some of the above cited works are limited in the attitude stability where other papers such as [6,8,10] treated the trajectory generation and tracking using the differential flatness approach.

In this work we introduce a new concept of the trajectory tracking problem by using a differential flatness–quaternion based approach. In order to follow the desired bath and assure the attitude stability a double loop control structure with the LQR control law is used.

This article is organized as follow: Section II gives a brief background over quaternion algebra and the dynamic modeling of quadrotors using quaternions, a linear model is than derived to be later used. Section III introduces the Differential flatness method and its application to the quadrotors model. Section IV develops the LQR control law. Section V shows and discusses the simulation results. Finally in Section VI the conclusion as well as the future recommendations is given.

III. QUATERNION MODELING

A. Quaternions Background

A quaternion, which belong to the quaternion space $\mathbb{H}$, is a hyper complex number of rank 4, which can be represented in many ways, but the one presented here is as a sum of a scalar component along an imaginary vector:[11]

$$q \in \mathbb{H}; \bar{q} \in \mathbb{R}^3; q_0 \in \mathbb{R}$$

$$q = q_0 + \bar{q} = q_0 + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad (1)$$

Where q is a given quaternion, $\bar{q}$ is the complex vectorial part and q_0 is the scalar part of q.

The main operation in quaternion algebra is the quaternion product, which is defined as: [11]

$$r \in \mathbb{H}; \bar{r} \in \mathbb{R}^3; r_0 \in \mathbb{R}$$

$$q \otimes r = (q_0 r_0 - \bar{q}.\bar{r}) + (r_0 \bar{q} + q_0 \bar{r} + \bar{q} \times \bar{r}) \quad (2)$$

The quaternion conjugate q^* is defined as: $q^* = q_0 - \bar{q}$.
The quaternion norm, $\|q\| \in \mathbb{R}$, is defined as:

$$\|q\| = \sqrt{q \otimes q^*} = \sqrt{{q_0}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2} \quad (3)$$

The quaternion inverse is obtained from $q^{-1} = \frac{q^*}{\|q\|}$ this work, only unit quaternions are used for the attitude representation, thus, $q^{-1} = q^*$. [11]

The three-dimensional rotation of any vector is given as a quaternion multiplication on the left by the unit quaternion q and on the right by its conjugate q^*. This mathematical operation can be rewritten as a multiplication of the matrix R_q and the abovementioned vector. Because we are rotating a vector, only the vector part from the rotation matrix is extracted, as can be seen in Eq. (4), where $q_{13\times}$ stands for a skew-symmetric matrix. The rotation matrix R_q is orthogonal; therefore the expression ${R_q}^{-1} = {R_q}^T$ is true. [3]

$$R_q = (q_0 I + q_{13\times})^2 + \bar{q}\bar{q}^T$$

$$= (q_0{}^2 - \bar{q}^T\bar{q})\, I + 2q_0 q_{13\times} + 2\bar{q}\bar{q}^T$$

$$= \begin{bmatrix} q_0{}^2 + q_1{}^2 - q_2{}^2 - q_3{}^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & q_0{}^2 - q_1{}^2 + q_2{}^2 - q_3{}^2 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & q_0{}^2 - q_1{}^2 - q_2{}^2 + q_3{}^2 \end{bmatrix} \quad (4)$$

The rotation can also be represented using a rotation vector as denoted in Eq. (5), where $\boldsymbol{u}$ is the rotation axis (unit vector) and α is the angle of rotation. Using this notation can have many benefits when creating an error or specifying a reference as it has a direct physical connection. [3]

$$q = \cos\frac{\alpha}{2} + u\sin\frac{\alpha}{2} \quad (5)$$

The derivative of a quaternion is given by the quaternion multiplication of the quaternion q and the angular velocity of the system ω, which in this case is the quadrotor. [3]

$$\dot{q} = \frac{1}{2} q \otimes \omega \quad (6)$$

Finally, for representing quaternion rotations in a more intuitive manner, the conversion from Euler angles to quaternion and from quaternion to Euler angle can be performed by utilizing the following two equations respectively. This property is very useful in case that the aim is to represent an orientation in angles, while retaining the overall dynamics of the system in a quaternion form. [4]

$$q = \begin{bmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{bmatrix}$$

$$\begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} atan2(2(q_0q_1 - q_2q_3), q_0{}^2 - q_1{}^2 - q_2{}^2 + q_3{}^2) \\ \operatorname{asin}(2(q_0q_2 - q_3q_1)) \\ atan2(2(q_0q_3 + q_1q_2), q_0{}^2 + q_1{}^2 - q_2{}^2 - q_3{}^2) \end{bmatrix} \quad (7)$$

B. Dynamic System Model

The quadrotor is an under-actuated system with 4 control inputs and 6 degrees of freedom.

The approach used more than any other to create a mathematical model of a wide variety of systems is to use the Newton–Euler formulation. It is based on the balance of forces and torques. An alternative energy-based approach is using Euler–Lagrange equations.

In this article the Newton–Euler equations will be derived, while assuming that the quadrotor is a rigid body and the centre of gravity coincides with the body-fixed frame origin.

The dynamic model of a quadrotor using Newton-Euler equations with quaternions is described as follows: [11]

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \\ q \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{p} \\ q \otimes \frac{T}{m} \otimes q^* + \bar{g} \\ \frac{1}{2} q \otimes \omega \\ J^{-1}(\tau - \omega \times J\omega) \end{bmatrix} \quad (8)$$

Where $p \in \mathbb{R}^3$ and $\dot{p} \in \mathbb{R}^3$ are the position and velocity vectors with respect to the inertial frame, T defines the thrust vector generated by the quadrotor motors, m and $\bar{g}$ represent the vehicle's mass and gravity vector, respectively, q describes the quaternion that represents the vehicle orientation with respect to the inertial frame, J introduces the inertia matrix with respect to the body-fixed frame and $\tau = \tau_u + \tau_{ext}$, where τ_u and τ_{ext} are the input and external torques respectively, applied on the aerial vehicle in the body-fixed frame. [11]

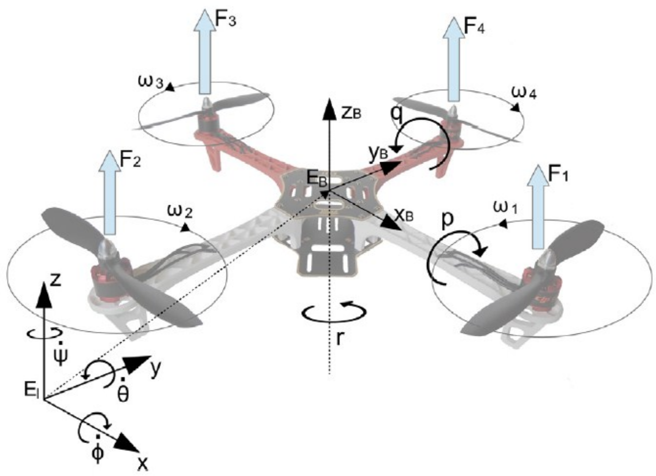

Fig. 1. Inertial and body-fixed frame of the quadrotor

The relationships between the input torques and forces is:

$$\begin{bmatrix} T \\ \tau_{u_x} \\ \tau_{u_y} \\ \tau_{u_z} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} k_i \omega_i{}^2 \\ l(k_1\omega_1{}^2 - k_2\omega_2{}^2 - k_3\omega_3{}^2 + k_4\omega_4{}^2) \\ l(k_1\omega_1{}^2 + k_2\omega_2{}^2 - k_3\omega_3{}^2 - k_4\omega_4{}^2) \\ \sum_{i=1}^{4} \tau_i(-1)^2 \end{bmatrix} \quad (9)$$

Where $k_i\omega_i{}^2$defines the thrust of the propeller of motor i with respect to its angular velocity ω_i, l is the distance from the center of mass to the motor axis of action and τ_i denotes the torque of motor i. [11]

The selected operating point is such the quadrotors is in hover, where : $\dot{p} = [0,0,0]^T$, $\omega = [0,0,0]^T$ and $q = [1,0,0,0]^T$ .Thus the dynamic equations can be linearized as follow:

$$J\dot{\omega} = \tau$$

$$\dot{q} = \frac{1}{2}\begin{bmatrix} 0 \\ \omega \end{bmatrix}$$

$$m\ddot{p} = \begin{bmatrix} mg(2q_0q_2) \\ -mg(2q_0q_2) \\ T \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = \begin{bmatrix} mg(u_x) \\ -mg(u_y) \\ T \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{10}$$

IV. DIFFERENTIAL FLATNESS

A system defined by the equation:

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$y = h(x(t)) \tag{11}$$

Where $x(t)$ is the state and $u(t)$ is the controller, is flat if there exists a vector $\big(x(t), u(t), u^{(1)}(t), u^{(2)}(t), \dots, u^{(\delta)}(t)\big)$, where the components are differentially independent, such that:

$$x(t) = \Xi_1(z(t), z^{(1)}(t), z^{(2)}(t), \dots, z^{(\alpha)}(t))$$
$$u(t) = \Xi_2\left(z(t), z^{(1)}(t), z^{(2)}(t), \dots, z^{(\beta)}(t)\right) \tag{12}$$

where α, β and δ are finite integers. Notations Ξ_1 and Ξ_2 represent two smooth maps, $z^{(i)}(t)$ is the ith derivative of $z(t)$. The vector $z(t)$ in the foregoing definition is called the flat output of the system.

By introducing the functions of Ξ_1 and Ξ_2, the flat output is composed by the variables which permit to parameterize all the other variables of the system.

Quadrotors have been shown to be a differentially at system with 4 at outputs. These at outputs are the inertial position of the vehicle, x, y, and z, and the yaw angle ψ. All of the quadrotor states can be written as a function of these four at outputs and the at output derivatives. These states include the position, velocity, and acceleration of the vehicle's center of mass, as well as the orientation, rotational velocity, and rotational acceleration of the vehicle. [12]

$$\mathcal{X} = \begin{bmatrix} x \\ y \\ z \\ \psi \end{bmatrix} \tag{13}$$

The mapping from the at outputs to the position, velocity, and acceleration of the quadrotors center of mass expressed in the inertial coordinate frame is trivial, as shown in Eq. (14).

$$[x, y, z]^T = [\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3]^T$$
$$[\dot{x}, \dot{y}, \dot{z}]^T = [\dot{\mathcal{X}}_1, \dot{\mathcal{X}}_2, \dot{\mathcal{X}}_3]^T \tag{14}$$

By manipulation of the equation of motion, the state vector and input vector can be expressed as a function of the output vector. [6]

$$\theta_d = \arctan(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) + \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g})$$

$$\varphi_d = \arctan(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) - \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g}.\cos\left(\arctan\left(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) + \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g}\right)\right))$$

$$\dot{\theta}_d = \left[\arctan(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) + \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g})\right]^{(1)}$$

$$\dot{\varphi}_d = \left[\arctan(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) - \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g}.\cos\left(\arctan\left(\frac{\ddot{\mathcal{X}}_1\cos(\mathcal{X}_4) + \ddot{\mathcal{X}}_2\sin(\mathcal{X}_4)}{\ddot{\mathcal{X}}_3 + g}\right)\right))\right]^{(1)} \tag{15}$$

Eq. (15).imply that the state x_i can be parameterized by variable $\mathcal{X}_i$ and its derivatives. Additionally, according to Eq. (9).the control input $u(t)$ is in terms of $z(t), \dot{z}(t)$, ω and $\dot{\omega}$. Therefore, the variable $\mathcal{X}$ is the flat output of system. The double loop control structure used in this work is shown in Fig.2.

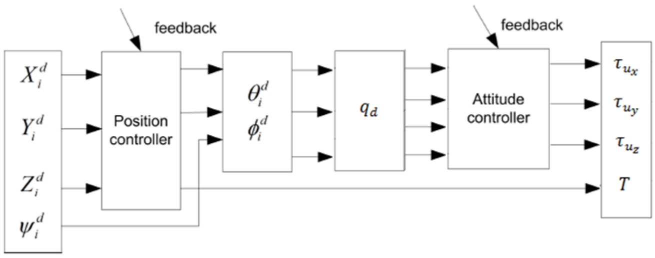

Fig. 2. Quadrotors double loop control structure

III. CONTROLLER DESIGN

Once the linearized model of the quadrotors in hover is derived given by Eq. (10), it can be written in the form of a state space system. The general formula of the state space representation is given by the following equation.

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{16}$$

The state space model of the simplified quadrotors attitude and position is given by Eqs. (17) and (18), respectively. The attitude state vector is $x_A = [\bar{q}\ \ \omega]$ and the position state vector is $x_P = [p\ \ \dot{p}]$. [3]

$$\dot{x_A} = \begin{bmatrix} 0_{3\times3} & 0.5I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} x_A + \begin{bmatrix} 0_{3\times3} \\ {I_q}^{-1} \end{bmatrix} \begin{bmatrix} \tau_{u_x} \\ \tau_{u_y} \\ \tau_{u_z} \end{bmatrix} \tag{17}$$

$$\dot{x_P} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} x_P + \begin{bmatrix} & 0_{3\times3} & \\ g & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} u_{xd} \\ u_{yd} \\ T_d \end{bmatrix} \tag{18}$$

The trajectory tracking controller consists of two parts, namely the position and the altitude controller.

The outer trajectory tracking controller is comprised of the altitude and the position controller as depicted in .Fig. 3. The output of the altitude controller is the desired total thrust T_d. The position controller generates the desired position quaternion value q_{pd} for the attitude controller. Note that the position quaternion will always have the 4th element equal to zero. [3]

$$q_{pd} = [q_{0pd}, q_{1pd}, q_{2pd}, 0]^T \tag{19}$$

The 4th element of the quaternion refers to a rotation around the z axis, which is given by the desired yaw angle and computed using expression (5), where the 2nd and 3rd elements of the quaternion will be zero, as we can see in Eq. (20).[3]

$$q_{zd} = [q_{0zd}, 0,0, q_{4zd}]^T \tag{20}$$

The desired quaternion for the attitude controller is then calculated as a combination of both rotations using the quaternion multiplication (2), which results in Eq. (21). [3]

$$q_d = q_{pd} \otimes q_{zd} \tag{21}$$

For both controllers the LQR command is used because it's a method to find the optimum solution for a problem of minimization that assures the system stability in close-loop, in addition its calculation is easy.

The Eq.(22). represents the quadratic cost function to minimize:

$$J(x,u) = \frac{1}{2}\int_0^{\infty} (x^T(t)Q_x(t) + u^T Ru(t))dt \tag{22}$$

R and Q are weight matrix used respectively in order to increase or to diminish the effect of the states and the entrances of individual form and are select for the designer in agreement with the required performance.

The optimum input is defined as U = -Lx With $L = R^{-1}B^T P$ and P is the solution to the Ricatti differential equation given by:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{23}$$

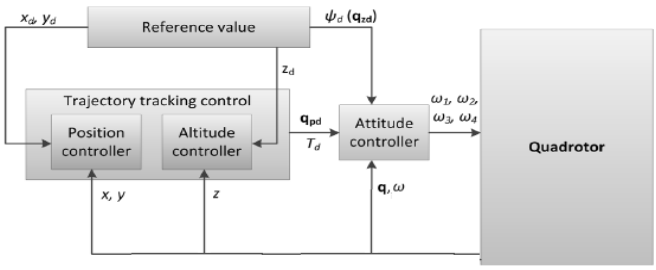

Fig. 3. Block diagram of the prposed control structure

IV. Simulations Results

The study case of the simulation is the tracking of a circular path with a 1m of diameter at a fixed altitude (1m) from a starting point $p_0 = (-1,0,0)$.

The gain matrix L in the inner and outer loop of the LQR controller was tuned to be adaptive with the full non-linear model described in Eq. (8).

The controller was simulated at a rate of 200 Hz which makes it suitable for a real implementation. All the necessary limitations over the actuators and the battery modeling were taking into consideration.

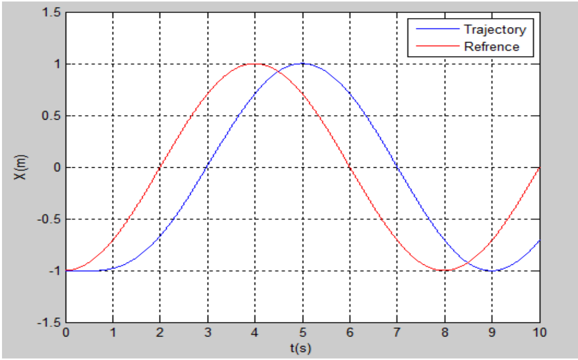

Fig. 4. X axes response

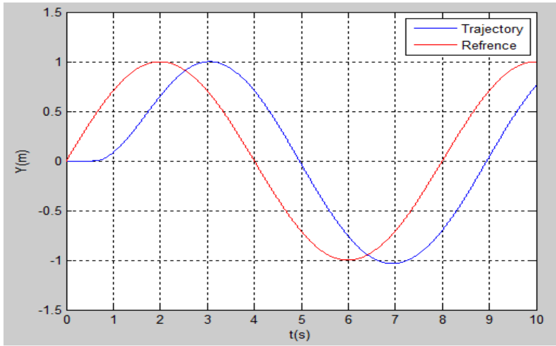

Fig. 5. Y axes response

Fig. 4,5,6 show the response obtained from the three coordinate tracking, we can see that the LQR controller has followed the desired consigns with a high accuracy, a delay of 1 sec in the X and Y axes is occurred due to the altitude settling time.

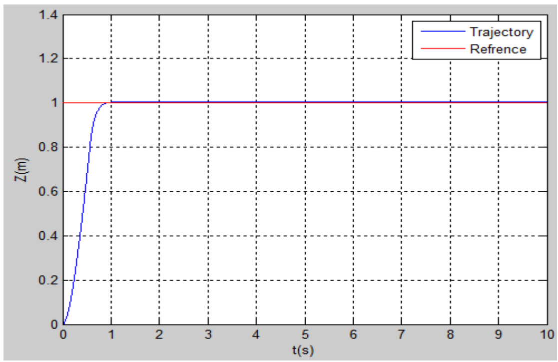

Fig. 6. Z axes response

The full trajectory is shown in Fig.7.

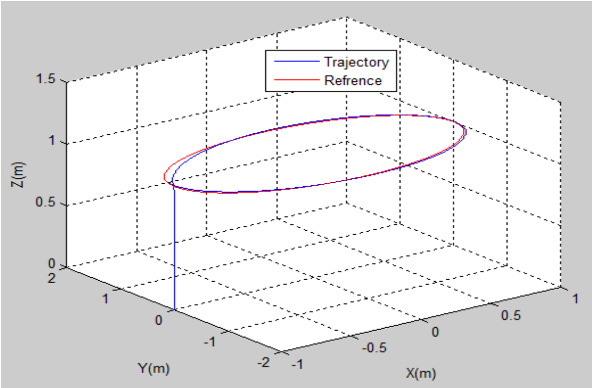

Fig. 7. Full trajectory tracking

Fig.8. introduces the input PMW control signals the four rotors, we notice that all the signals are within the range of 1-2 sec which correspond to the ESC (Electronic Speed Controller) functional rate.

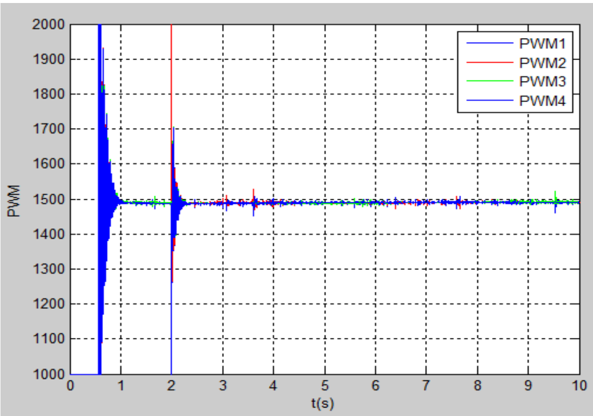

Fig. 8. PWM input control signals

The obtained quaternions used in the attitude stability are depicted in Fig.9. , it can be seen that stability is assured around the unit quaternion.

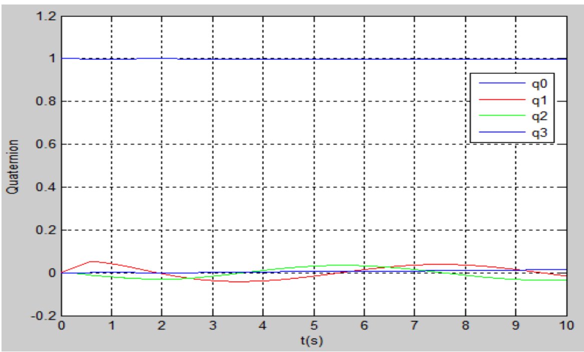

Fig. 9. Quternions used for attitude stability

I. Conclusion

In this paper the problem of the quadrotors trajectory tracking was studied using a differential flatness- quaternion based approach.

The obtained results were juged to be satisfactory since the quadrotors has successfully followed the desired path.

A double loop control structure based on the LQR command was applied in order to maintain the trajectory tracking and the attitude stability.

The use of the quaternion instead of Euler angles eliminate the gimball lock or any discontinuities that can occur.

Despite we have obtained good results we recommend the use of the non-linear control laws such as the backstepping control to make the system more robust.

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